SENSITIVITY ANALYSIS OF STEERING SYSTEM MODEL

Dariusz Zardecki

Military University of Technology, Faculty of Mechanical Engineering
Institute of Motor Vehicles and Transportation
Gen. S. Kaliskiego 2, 00-908 Warsaw, Poland
tel.+48 22 683 75 24, fax. +48 22 683 73 66
e-mail: dzardecki@wat.edu.pl

Abstract

The method of sensitivity analysis of mathematical model with strong non-linear characteristics is discussed in the paper. This model focuses on friction, as well as freeplay dynamics in a car steering system including stick-slip phenomena. Because of strong nonlinearities of the model, its sensitivity indexes can not be counted by classic sensitivity variation equations and classic mathematical analysis of smooth functions. Fortunately, nonlinear characteristics are described with using special piecewise - linear luz(…) and tar(…) projections. They have an original mathematical apparatus which secures parametrically made simplifications and enough regularity of the model (some details of analytical transformations are shown). Thanks to luz(…) and tar(…) projections the continuity of sensitivity indexes is secured but of course, the calculation of sensitivity indexes has to base on comparative simulation investigations. The paper presents example results of simulations of the steering system action (including stick-slip processes) for different values of friction as well as freeplay parameters (kinetic and static friction in king-pins, backlash in a steering gear). These simulations concern combined open road tests of a passenger car according to the ISO requirements (ISO 7401 – the ramp input on the steering wheel, then ISO 4138 – circular steady state motion, and finally the steering wheel released, all with a constant speed 80 km/h).

Keywords: *car, steering system, piecewise-linear model, simulation, sensitivity analysis, friction, freeplay, stick-slip*

1. Introduction

Every car steering system can be treated as a nonlinear multi-body mechanism. Because of a dry friction (with stiction, especially in king-pins) and a freeplay (well known as a backlash in steering gear) the steering system can be a source of singular dynamic effects (non-linear vibrations, stick-slip phenomena, angular off-set of steered wheels after steering wheel release). Such dynamic processes influent on handling, steerability, and generally on active safety of a car. Indeed, the steering system friction and freeplay are examined strictly when a car passes seasonal diagnostic tests. Too high freeplay and non-standard friction in a steering mechanism are unacceptable. But on the other hand, these attributes are rarely taken into account in theoretical and simulation studies. Sporadic examples of somebody else papers focused on friction or freeplay problems in car steering systems are discussed in the Zardecki's monograph [14], and recently in the paper [5]. By the way, the paper [5] contains collective information on the resistance force (friction) and angular dead-zone (freeplay) parameters (results of stand tests in the PIMOT) of many passenger cars, busses, and trucks. They show that friction/freeplay parameters might be surprisingly very diverse (and high) even in new cars. Therefore problems of steering system non-linear dynamics (i.e. dynamics including friction forces in kingpins and freeplay of steering gear) seem to be important for practice and very interesting for science.

Modern sophisticated studies of dynamical systems base on mathematical models, on their sensitivity analysis, and finally on simulation investigations. These three trends have been present in author's scientific works, but most of his papers were devoted to mathematical models and simulation investigations. The background of modelling – special piecewise-linear luz(…) and tar(…) projections with their mathematical apparatus has been shown in [10], [11], [14].

Derivations of elementary friction/freeplay models have presented in [12], [13], [14]. Synthesis of the models of steering system as well as their reductions to simplified forms is described in the monograph [14]. Several papers, for example [5], [6], [7], [8], present results of simulation studies of a car lateral dynamics including friction and freeplay effects.

This paper deals with problems (especially theoretical problems) of sensitivity analysis of steering system models including friction/freeplay submodels. Such models use strong non-linear characteristics (even for small excitation), and have variable-structure form (stick-slip phenomenon). So, from theoretical point of view the models of systems working with friction and/or freeplay must be classified as non-smooth models.

2. Theoretical background of sensitivity analysis of non-smooth models

Sensitivity analysis of mathematical models concerns calculations of measures (e.g. integral indexes) of differences between signals from nominal and change models.

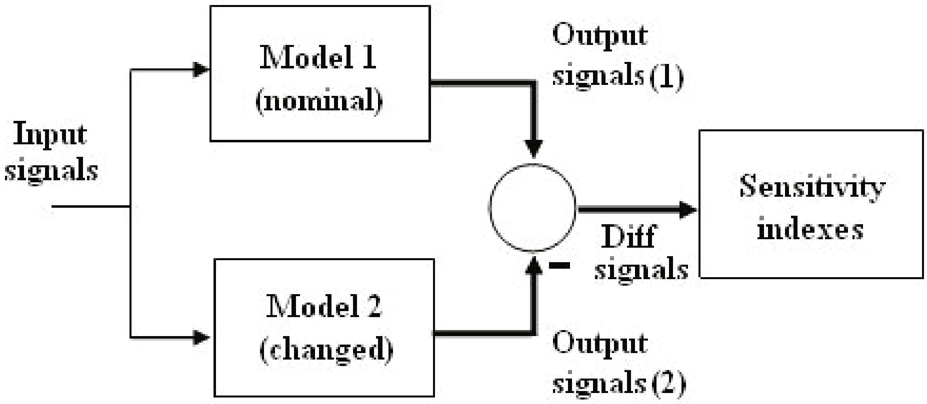

Fig. 1. General schematic diagram of sensitivity analysis

When model equations have regular forms and their changes result from parametrically made perturbations the sensitivity indexes are continuous functions of these parameters. Classic parametric methods take advantage of a variation analysis, so the continuity and differentiation of model equations is demanded (Franc [2]).

In case of models of mechanisms working with friction and freeplay (non-smooth models), conditions for differentiation of model equations are not fulfilled. The question appears: What are the conditions of “regularity” of the model for assertion of continuity of sensitivity indexes when model parameter is changed. According to classic analysis of ordinary differential equations (model equations) their variables are continuous in relation to their non-singular parameters of equations. The matter is more complicated for parameters that steer a structure of the model and cause a reduction of a high-order model to its low-order form (singular perturbation problem).

Let consider a singular perturbation of a high-order dynamical system model (1):

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{y}, \varepsilon, t), \quad \underline{x}(t_0) = \underline{x}_0, \ \underline{x} \in X \subset R^n,$$

$$\varepsilon \dot{\underline{y}} = \underline{h}(\underline{x}, \underline{y}, \varepsilon, t), \quad \underline{y}(t_0) = \underline{y}_0, \ \underline{y} \in Y \subset R^m, \ t \geq t_0 \in T \subset R^1_+, \ \varepsilon \geq 0 \in E \subset R^1_+.$$

For $\varepsilon \to 0$, the singularly-perturbed low-order model (2) is obtained:

$$\dot{\tilde{\underline{x}}} = \underline{f}(\tilde{\underline{x}}, \tilde{\underline{y}}, 0, t), \quad \tilde{\underline{x}}(t_0) = \underline{x}_0, \ \tilde{\underline{x}} \in X \subset R^n,$$

$$\underline{0} = \underline{h}(\tilde{\underline{x}}, \tilde{\underline{y}}, 0, t), \quad \tilde{\underline{y}}(t_0) = \underline{y}_0, \ \tilde{\underline{y}} \in Y \subset R^m, \ t \geq t_0 \in T \subset R^1.$$

Now, the reduced model consists of differential and degenerate implicit function equations which are constraints for dynamical part of the model. Of course, a simulation of such mixed subsystems can not be easy because of necessity of iterations. The problem seems to be especially difficult when both forms of the model have a non-compatibility of their initial conditions. Generally, $\underline{0} \neq \underline{h}(x_0, y_0, 0, t_0)$ (3) is possible! In this study we assume that initial conditions are well-defined i.e. $\underline{0} = \underline{h}(x_0, y_0, 0, t_0)$ (4).

Transformation of singularly-perturbed high-order model to its low-order form is well-defined (regular) when for $t \geq t_0$:

$$\lim_{\varepsilon \to 0} \underline{x}(t) = \underline{\tilde{x}}(t), \quad \lim_{\varepsilon \to 0} \underline{y}(t) = \underline{\tilde{y}}(t). \tag{5}$$

The detail question is: What should be the right-hand functions in the high-order model for assure regularity of its reduction? For answer, the Tikhonov theory ([9]) "on the dependence of solutions on a small parameter" seems to be the most convenient. According to Tikhonov theorem, when initial conditions are well-defined, for regular parametrical reduction the model should fulfil the following two assumptions:

There is the function $\underline{\tilde{y}} = \underline{g}(\underline{\tilde{x}}, t)$ which fulfils $\underline{0} = \underline{h}(\underline{\tilde{x}}, \underline{g}(\underline{\tilde{x}}, t), 0, t)$. (6)

The solution $\underline{\bar{y}}(\tau)$ of the equations $\frac{d\underline{\bar{y}}(\tau)}{d\tau} = \underline{h}(\underline{\tilde{x}}, \underline{\bar{y}}(\tau), 0, \tau)$ with $\underline{\bar{y}}(\tau = t_0) = \underline{y}_0$. (7)

Fulfils $\lim_{\tau \to +\infty} \underline{\bar{y}}(\tau) = \underline{g}(\underline{\tilde{x}}, t)$ (asymptotic stability) in $X \circ Y \circ T$. (8)

So we can formulate several conclusions for sensitivity of non-smooth models:

- Classic sensitivity analysis basing on variation equations and variation indexes must be replaced by simulation investigations repeated for different values of parameters.
- Efficiency of such studies demands regular models without implicit algebraic forms.
- The regularity of non-smooth system model with well-defined initial conditions and stable degenerated equations occurs if variables of degenerated equations can be extracted.
- Unravelling of the reduced equations appears as the most important problem for theoretical structural sensitivity studies.
- Analytical forms of friction/freeplay submodels are necessary for parametric formulation of sensitivity analysis. They should assure unravelling of degenerate equations.

Piecewise linear luz(…) and tar(…) projections with their special mathematical apparatus appear as a very efficient method for solving these description problems important for sensitivity analysis of steering system models including friction/freeplay components.

3. The luz(…) and tar(….) projections in friction/freeplay models

General methods of modelling of multi-body systems with friction (kinetic and static) and freeplay (backlash, clearance) provide strong non-linear models with constrains (Brogliato [1], Grzesikiewicz [3]). Certainly such models are very difficult for theoretical analysis of stick-slip phenomena as well as for analytical model reductions. They are also onerous in simulation programs because of iterative procedures. So, a quest of a more "user-friendly" method of modelling of multi-body systems has ever been very attractive and appreciated scientific challenge. Therefore, semi-analytical Karnopp's models [4] of stick-slip phenomena in two-mass systems (very useful for modelling multi-body serial systems treated as an aggregation of multiple single-mass or double-mass subsystems) are cited by researchers many times. Note that steering mechanisms might be partially treated as serial systems.

In cases of steering mechanisms, the friction and freeplay actions can be expressed by piecewise linear models basing on piecewise linear characteristics, see Fig. 2.

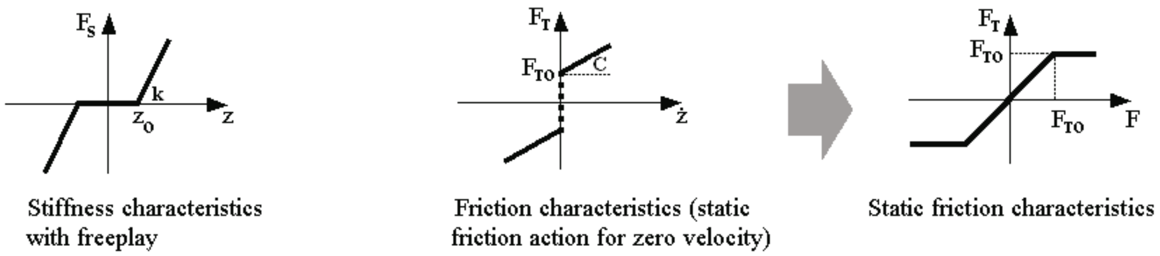

Fig. 2. Typical characteristics for freeplay and friction description. Here version when for kinetic and static friction force parameter $F_{TK0} = F_{TS0} = F_{T0}$. Notation: k – stiffness coefficient, C – viscous friction coefficient , F_{T0} – dry friction parameter, F_S – stiffness force, F_T – Friction force, z – displacement, $\dot{z}$ – velocity, F – acting force

Such characteristics can be described analytically with using simple piecewise-linear luz (…) and tar(...) projections.

Definition:

For $-\infty < x < +\infty$ and $a \geq 0$:

$$\mathrm{luz}(x,a) = x + \frac{|x-a| - |x+a|}{2}, \tag{9}$$

$$\mathrm{tar}(x,a) = x + a \cdot \mathrm{sgh}(x), \quad \text{where} \quad \mathrm{sgh}(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ s^* \in [-1,1] & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}. \tag{10}$$

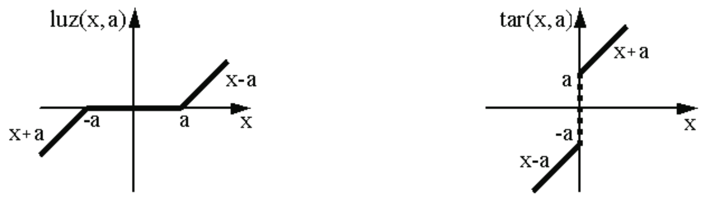

Fig. 3. Topology of luz (…) and tar(…) projections

Example spectacular formulas and theorems are listed below. They are especially useful for modelling of steering system dynamics and sensitivity analysis. All constants are non-negative.

$$\mathrm{luz}(x,a) = \mathrm{tar}^{-1}(x,a) \qquad \mathrm{tar}(x,a) = \mathrm{luz}^{-1}(x,a) \qquad \text{(inversion formulas)}, \tag{11,12}$$

$$\mathrm{luz}(-x,a) = -\mathrm{luz}(x,a) \qquad \mathrm{tar}(-x,a) = -\mathrm{tar}(x,a), \tag{13,14}$$

$$k \cdot \mathrm{luz}(x,a) = \mathrm{luz}(k \cdot x, k \cdot a) \qquad k \cdot \mathrm{tar}(x,a) = \mathrm{tar}(k \cdot x, k \cdot a), \tag{15,16}$$

$$\mathrm{luz}(\mathrm{luz}(x,a), b) = \mathrm{luz}(x, a+b), \qquad \mathrm{tar}(\mathrm{tar}(x,a), b) = \mathrm{tar}(x, a+b), \tag{17,18}$$

$$k_1 \cdot \mathrm{tar}(x,a_1) + k_2 \cdot \mathrm{tar}(x,a_2) = (k_1 + k_2) \cdot \mathrm{tar}\left(x, \frac{k_1 \cdot a_1 + k_2 \cdot a_2}{k_1 + k_2}\right). \tag{19}$$

If $\mathrm{luz}(y,b) = k \cdot \mathrm{luz}(x-y, a)$,

then $\mathrm{luz}(y,b) = \frac{k}{k+1} \cdot \mathrm{luz}(x, a+b)$ (unravelling formula) . (20)

If $\dot{x}(t) \in y(t) - b \cdot \mathrm{tar}(x(t), a)$ and $\mathrm{tar}(0,a): \min_{\mathrm{tar}(0,a)} Q(\dot{x}) \wedge \mathrm{tar}(0,a) \in [-a, a]$,

where $Q(...)$ – convex function,

$$\text{then} \quad \dot{x}(t) = \begin{cases} y(t) - b \cdot \mathrm{tar}(x(t), a) & \text{if} \quad x(t) \neq 0 \\ \mathrm{luz}(y(t), b \cdot a) & \text{if} \quad x(t) = 0 \end{cases}. \tag{21}$$

If $\varepsilon \cdot \dot{x}(t) \in y(t) - b \cdot tar(x(t), a)$ and $\varepsilon \to 0$,

Then
$$x(t) = luz\left(\frac{y(t)}{b}, a\right). \tag{22}$$

Simple analytical forms of the formulas are the main advantages of this mathematical apparatus. Some formulas have semi-linear forms! The luz (…) and tar(…) projections can be used to describe different piecewise linear characteristics in compact analytic forms.

Characteristics presented on Fig.2 can be expressed analytically:

$$F_s = k\, luz(z, z_0) \tag{23}$$

$$F_T = \begin{cases} C\, tar\left(\dot{z}, \frac{F_{T0}}{C}\right) & if \quad \dot{z} \neq 0 \\ F - luz(F, F_{T0}) & if \quad \dot{z} = 0. \end{cases} \tag{24}$$

The piecewise-linear analytical forms with luz(…) and tar(…) projections occur in elementary stick-slip models too. The elementary stick-slip models for single-mass and double-mass systems presented below have been derived in the papers [13], [14].

Stick-slip model for single-mass system:

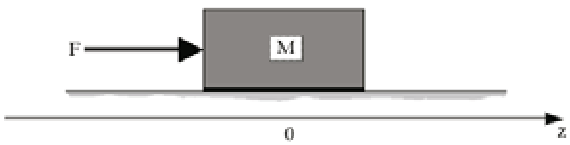

Fig. 4. Single-mass system with friction

$$M \cdot \ddot{z}(t) = \begin{cases} F(t) - C \cdot tar\left(\dot{z}(t), \frac{F_{T0}}{C}\right) & if \quad \dot{z}(t) \neq 0 \\ luz(F(t), F_{T0}) & if \quad \dot{z}(t) = 0. \end{cases} \tag{25}$$

Note: When $\dot{z}(t) = 0$ and $|F(t)| \leq F_{T0}$ we obtain $luz(F(t), F_{T0}) = 0$, so also $\ddot{z}(t) = 0$ (stick state). When $|F(t)| > F_{T0}$, we have $luz(F(t), F_{T0}) \neq 0$ and $\ddot{z}(t) \neq 0$ (slip state).

Note: When M → 0 (reduction of the model), after inversion of tar(…) we obtain uninvolved form $C \cdot \dot{z}(t) = luz(F(t), F_{T0})$ (26) (no motion for $-F_{T0} \leq F(t) \leq F_{T0}$)).

Stick-slip model for double-mass system:

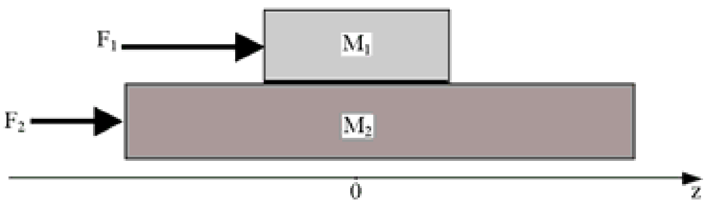

Fig. 5. Double-mass system with friction

$$M_1 \cdot \ddot{z}_1(t) = \begin{cases} F_1(t) - C \cdot tar\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T0}}{C}\right) & if \quad \dot{z}_1(t) \neq \dot{z}_2(t) \\ \frac{M_1}{M_1 + M_2} \cdot (F_1(t) + F_2(t)) + luz\left(\frac{M_2 \cdot F_1(t) - M_1 \cdot F_2(t)}{M_1 + M_2}, F_{T0}\right) & if \quad \dot{z}_1(t) = \dot{z}_2(t), \end{cases} \tag{27}$$

$$M_2 \cdot \ddot{z}_2(t) = \begin{cases} F_1(t) + C \cdot tar\left((\dot{z}_1(t) - \dot{z}_2(t)), \frac{F_{T0}}{C}\right) & if \quad \dot{z}_1(t) \neq \dot{z}_2(t) \\ \frac{M_2}{M_1 + M_2} \cdot (F_1(t) + F_2(t)) - luz\left(\frac{M_2 \cdot F_1(t) - M_1 \cdot F_2(t)}{M_1 + M_2}, F_{T0}\right) & if \quad \dot{z}_1(t) = \dot{z}_2(t). \end{cases} \tag{28}$$

Note: When $\dot{z}_1(t) = \dot{z}_2(t)$ and $-F_{T0} \le \frac{M_2 \cdot F_1(t) - M_1 \cdot F_2(t)}{M_1 + M_2} \le F_{T0}$, then

$$(M_1 + M_2)\cdot \ddot{z}_1(t) = F_1(t) + F_2(t) \qquad \text{or} \qquad (M_1 + M_2)\cdot \ddot{z}_2(t) = F_1(t) + F_2(t)\,. \tag{29, 30}$$

These equations have identical forms. It means that $\ddot{z}_1(t) = \ddot{z}_2(t)$ (stick state).

Note: When $M_2 \to \infty$, the state $\ddot{z}_2(t) = 0$ must be steady. It means a blockade of this block. So also $\dot{z}_2(t) = 0$. After reduction the model passes to the single-mass (M_1) system model.

We can formulate conclusions important for sensitivity analysis and simulation studies of non-smooth stick-slip models basing on luz (…) and tar(…) projections:

- The elementary stick-slip models are regular during parametrically made reductions.
- These models can be applied directly with standard numerical procedures.

4. Regularity of steering system models basing on luz(…) and tar(…) projections

Let consider multi-body rotary system as substitutive system of real steering system mechanism. This physical model contains not only main elements, but also supplementary mass-less gear wheels and infinitely large stiffness of shafts that facilitate a synthesis of equations of motion. The freeplay concerns the gearbox tooth backlash. The friction elements (with stiction action) concern the kingpin bearings as well as the vibration damper.

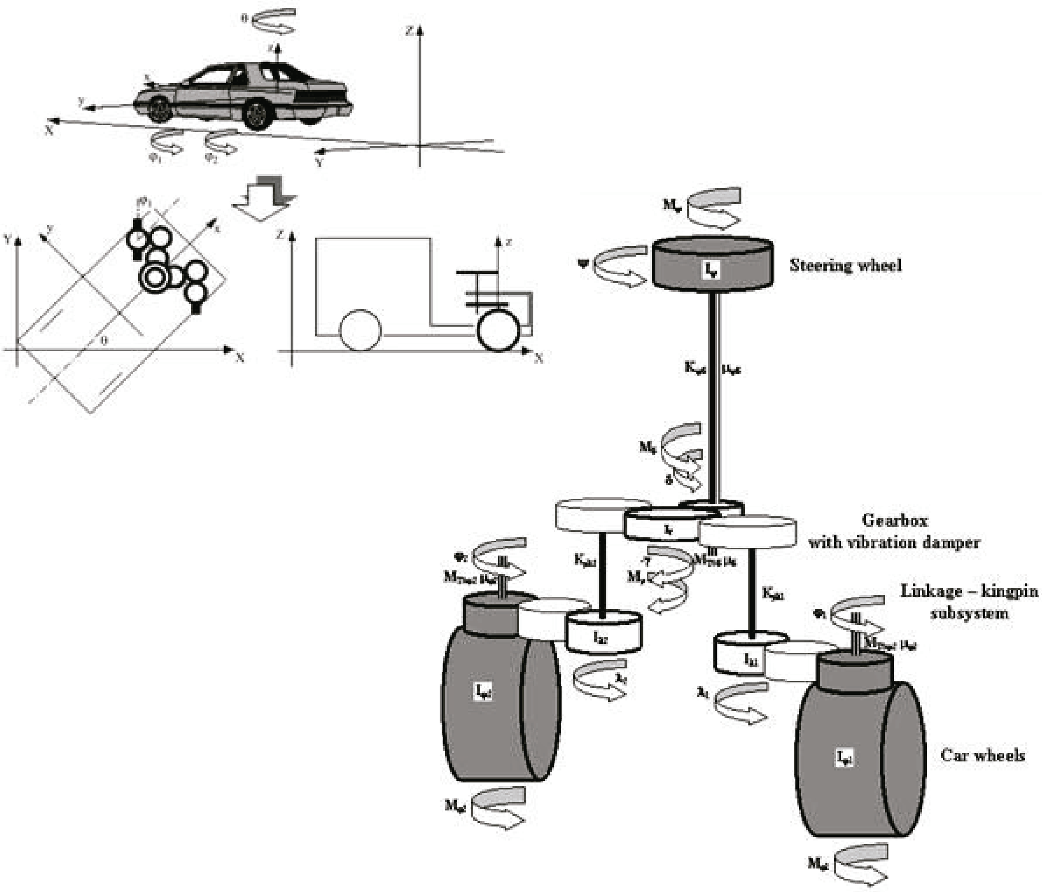

Fig. 6. Idea of substitutive physical model of steering system mechanism

Notation:

ψ - steering wheel angle

δ, γ - steering gear input and output angle

λ_1, λ_2 - steering linkage substitutive elements' output angles
φ_1, φ_2 - kingpin (and steered wheels') angles
M_ψ - steering wheel input torque
M_δ, M_γ - steering gear assistance torques (two cases)
$M_{\varphi1}, M_{\varphi2}$ - kingpin external torques (wheels stabilization plus unbalance)
I_ψ - moment of inertia of steering wheel with steering column
I_δ, I_γ - moment of inertia of gearbox input and output wheel
$I_{\lambda1}, I_{\lambda2}$ - moments of inertia of steering linkage substitutive elements
$I_{\varphi1}, I_{\varphi2}$ - moments of inertia of kingpins with steered wheels
$K_{\psi\delta}$ - steering column stiffness coefficient
$K_{\delta\gamma}$ - gearbox teeth stiffness coefficient
$K_{\gamma\lambda1}, K_{\gamma\lambda2}$ - stiffness coefficients of substitutive linkage shafts
$K_{\lambda1\varphi1}, K_{\lambda2\varphi2}$ - stiffness coefficients of substitutive linkage-kingpin gear subsystem
$\mu_{\psi\delta}$ - steering column's substitutive material damping coefficient
μ_δ - damping coefficient of steering mechanism damper
$\mu_{\varphi1}, \mu_{\varphi2}$ - damping coefficient of kingpin bearings
$M_{T0\delta}$ - maximal dry friction torque of gearbox damper
$M_{T0\varphi1}, M_{T0\varphi2}$ - maximal dry friction torques of king pin bearings
$(\delta - p\gamma)_0$ - gear freeplay parameter (1/2 of freeplay seen from steering wheel)
p - gear ratio (for steady state $\delta = p \cdot \gamma$)
n_1, n_2 - substitutive gear ratios of left and right part of steering linkage - kingpin subsystem (for steady states $\lambda_1 = n_1 \cdot \varphi_1$, $\lambda_2 = n_2 \cdot \varphi_2$). They can be nonlinear functions of φ_1 and φ_2.

In consideration of some parameters' disproportions we will set in utility models: $I_\delta = 0$, $I_\gamma = 0$, $I_{\lambda1} = 0$, $I_{\lambda2} = 0$, $K_{\lambda1\varphi1} \to \infty$, $K_{\lambda2\varphi2} \to \infty$, $K_{\delta\gamma} \to \infty$.

Detail derivation of mathematical model and its simplified forms is present in the monograph [14]. Here, only the main transformations important for regularity of the model are shown.

The primary mathematical model is based on the standard Newton's equations. For shorten description, detail formulas concerning static friction states are not presented for the moment.

$$I_\psi \cdot \ddot{\psi} + \mu_{\psi\delta} \cdot (\dot{\psi} - \dot{\delta}) + K_{\psi\delta} \cdot (\psi - \delta) = M_\psi , \tag{31}$$

$$I_\delta \cdot \ddot{\delta} + \mu_\delta \cdot \mathrm{tar}\left(\dot{\delta}, \frac{M_{T0\delta}}{\mu_\delta}\right) - \mu_{\psi\delta} \cdot (\dot{\psi} - \dot{\delta}) - K_{\psi\delta} \cdot (\psi - \delta) + K_{\delta\gamma} \cdot \mathrm{luz}(\delta - p \cdot \gamma, (\delta - p\gamma)_0) = M_\delta , \tag{32}$$

$$I_\gamma \cdot \ddot{\gamma} - p^2 \cdot K_{\delta\gamma} \cdot \mathrm{luz}\left(\frac{\delta}{p} - \gamma, \frac{(\delta - p\gamma)_0}{p}\right) + K_{\gamma\lambda1} \cdot (\gamma - \lambda_1) + K_{\gamma\lambda2} \cdot (\gamma - \lambda_2) = M_\gamma , \tag{33}$$

$$I_{\lambda1} \cdot \ddot{\lambda}_1 - K_{\gamma\lambda1} \cdot (\gamma - \lambda_1) + K_{\lambda1\varphi1} \cdot (\lambda_1 - n_1 \cdot \varphi_1) = 0 , \tag{34}$$

$$I_{\lambda2} \cdot \ddot{\lambda}_2 - K_{\gamma\lambda2} \cdot (\gamma - \lambda_2) + K_{\lambda2\varphi2} \cdot (\lambda_2 - n_2 \cdot \varphi_2) = 0 , \tag{35}$$

$$I_{\varphi1} \cdot \ddot{\varphi}_1 + \mu_{\varphi1} \cdot \mathrm{tar}\left(\dot{\varphi}_1, \frac{M_{T0\varphi1}}{\mu_{\varphi1}}\right) - n_1^2 \cdot K_{\lambda1\varphi1} \cdot \left(\frac{\lambda_1}{n_1} - \varphi_1\right) = M_{\varphi1} , \tag{36}$$

$$I_{\varphi2} \cdot \ddot{\varphi}_2 + \mu_{\varphi2} \cdot \mathrm{tar}\left(\dot{\varphi}_2, \frac{M_{T0\varphi2}}{\mu_{\varphi2}}\right) - n_2^2 \cdot K_{\lambda2\varphi2} \cdot \left(\frac{\lambda_2}{n_2} - \varphi_2\right) = M_{\varphi2} . \tag{37}$$

When $I_\gamma, I_\delta = 0$, $K_{\lambda1\varphi1}, K_{\lambda2\varphi2} \to \infty$ the primary model is reduced (parametric transformation) to:

$$I_\psi \cdot \ddot{\psi} + \mu_{\psi\delta} \cdot (\dot{\psi} - \dot{\delta}) + K_{\psi\delta} \cdot (\psi - \delta) = M_\psi , \tag{38}$$

$$\dot{\delta}=\frac{1}{\mu_{\delta}+\mu_{\psi\delta}}\mathrm{luz}\left(\mu_{\psi\delta}\cdot\dot{\psi}+K_{\psi\delta}\cdot(\psi-\delta)-K_{\delta\gamma}\cdot\mathrm{luz}\left(\delta-p\cdot\gamma,(\delta-p\gamma)_0\right)+M_{\delta},M_{T0\delta}\right),\quad (\mathrm{tar}^{-1}(\ldots)\text{ was used}) \tag{39}$$

$$-p^2\cdot K_{\delta\gamma}\cdot\mathrm{luz}\left(\frac{\delta}{p}-\gamma,\frac{(\delta-p\gamma)_0}{p}\right)+K_{\gamma\lambda1}\cdot(\gamma-n_1\cdot\varphi_1)+K_{\gamma\lambda2}\cdot(\gamma-n_2\cdot\varphi_2)=M_{\gamma}\text{ , (algebraic equation)} \tag{40}$$

$$I_{\varphi1}\cdot\ddot{\varphi}_1+\mu_{\varphi1}\cdot\mathrm{tar}\left(\dot{\varphi}_1,\frac{M_{T0\varphi1}}{\mu_{\varphi1}}\right)-n_1\cdot K_{\gamma\lambda1}\cdot(\gamma-n_1\cdot\varphi_1)=M_{\varphi1}\,, \tag{41}$$

$$I_{\varphi2}\cdot\ddot{\varphi}_2+\mu_{\varphi2}\cdot\mathrm{tar}\left(\dot{\varphi}_2,\frac{M_{T0\varphi2}}{\mu_{\varphi2}}\right)-n_2\cdot K_{\gamma\lambda2}\cdot(\gamma-n_2\cdot\varphi_2)=M_{\varphi2}\,. \tag{42}$$

For regularity of the model, the γ should be removed. So, our algebraic equation is rewritten as

$$\gamma-\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}=\frac{p^2K_{\delta\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\cdot\mathrm{luz}\left(\left(\frac{\delta}{p}-\gamma\right),\frac{(\delta-p\gamma)_0}{p}\right) \tag{43}$$

That is

$$\begin{aligned}&\mathrm{luz}\left(\left(\gamma-\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),0\right)=\\&=\frac{p^2K_{\delta\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\mathrm{luz}\left(\left(\frac{\delta}{p}-\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}-\left(\gamma-\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right)\right),\frac{(\delta-p\gamma)_0}{p}\right)\end{aligned}. \tag{44}$$

Applying the unravelling formula (see p.3) we can extract the γ

$$\gamma=\frac{1}{1+\dfrac{K_{\gamma\lambda1}+K_{\gamma\lambda2}}{p^2K_{\delta\gamma}}}\cdot\mathrm{luz}\left(\left(\frac{\delta}{p}-\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),\frac{(\delta-p\gamma)_0}{p}\right)+\frac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\,. \tag{45}$$

Setting $K_{\delta\gamma}\to\infty$, after simple transformations we obtain a final variable-structure form respecting kinetic as well as static friction states (necessary for stick-slip description):

$$I_{\psi}\ddot{\psi}+\mu_{\psi\delta}(\dot{\psi}-\dot{\delta})+K_{\psi\delta}(\psi-\delta)=M_{\psi} \tag{46}$$

$$\dot{\delta}=\frac{1}{\mu_{\delta}+\mu_{\psi\delta}}\mathrm{luz}\left(\begin{pmatrix}\mu_{\psi\delta}\dot{\psi}+K_{\psi\delta}(\psi-\delta)\\-\dfrac{K_{\gamma\lambda1}+K_{\gamma\lambda2}}{p^2}\mathrm{luz}\left(\left(\delta-p\dfrac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),(\delta-p\gamma)_0\right)+M_{\delta}\end{pmatrix},M_{T0\delta}\right), \tag{47}$$

$$I_{\varphi1}\ddot{\varphi}_1=\begin{cases}M_{\varphi1}-\mu_{\varphi1}tar\left(\dot{\varphi}_1,\dfrac{M_{T0\varphi1}}{\mu_{\varphi1}}\right)+n_1K_{\gamma\lambda1}\begin{pmatrix}luz\left(\left(\dfrac{\delta}{p}-\dfrac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),\dfrac{(\delta-p\gamma)_0}{p}\right)+\\+\dfrac{K_{\gamma\lambda2}(n_2\varphi_2-n_1\varphi_1)+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\end{pmatrix} & if\quad \dot{\varphi}_1\neq0\\ luz\left(\left(n_1K_{\gamma\lambda1}\begin{pmatrix}luz\left(\left(\dfrac{\delta}{p}-\dfrac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),\dfrac{(\delta-p\gamma)_0}{p}\right)+\\+\dfrac{K_{\gamma\lambda2}(n_2\varphi_2-n_1\varphi_1)+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\end{pmatrix}+M_{\varphi1}\right),M_{T0\varphi1}\right) & if\quad \dot{\varphi}_1=0,\end{cases} \tag{48}$$

$$I_{\varphi2}\ddot{\varphi}_2=\begin{cases}M_{\varphi2}-\mu_{\varphi2}tar\left(\dot{\varphi}_2,\dfrac{M_{T0\varphi2}}{\mu_{\varphi2}}\right)+n_2K_{\gamma\lambda2}\begin{pmatrix}luz\left(\left(\dfrac{\delta}{p}-\dfrac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\right),\dfrac{(\delta-p\gamma)_0}{p}\right)+\\+\dfrac{K_{\gamma\lambda1}(n_1\varphi_1-n_2\varphi_2)+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\end{pmatrix} & if\quad \dot{\varphi}_2\neq0\\ luz\left(\left(n_2K_{\gamma\lambda2}\begin{pmatrix}luz\left(\dfrac{\delta}{p}-\dfrac{K_{\gamma\lambda1}n_1\varphi_1+K_{\gamma\lambda2}n_2\varphi_2+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}},\dfrac{(\delta-p\gamma)_0}{p}\right)+\\+\dfrac{K_{\gamma\lambda1}(n_1\varphi_1-n_2\varphi_2)+M_{\gamma}}{K_{\gamma\lambda1}+K_{\gamma\lambda2}}\end{pmatrix}+M_{\varphi2}\right),M_{T0\varphi2}\right) & if\quad \dot{\varphi}_2=0.\end{cases} \tag{49}$$

This form has not any algebraic loops and is convenient to standard ODE (Ordinary Differential Equations) procedures.

The regularity of the model means that it can be applied as the base model for simulation studies of friction/freeplay effects in steering system mechanism and for sensitivity analysis when friction/freeplay parameters are varied. Thanks to luz(…) and tar(…) projections the continuity of sensitivity indexes is secured. Of course, calculations of sensitivity indexes have to base on comparative simulations.

5. Simulation investigations and sensitivity conclusions

The presented regular model was used in extensive simulation investigations. Those studies were focused on the sensitivity of the car dynamics when friction or/and freeplay parameters are varied.

For sensitivity analysis the FORS program (special simulation program) was elaborated. This program enables simulation investigations of passenger car which is tested according to several ISO and ECE regulations. The model of car dynamics contains two main submodels – the partial model of the steering system dynamics and the partial model of the vehicle motion. These submodels are very complex, and have modular forms (Fig. 7). They allow the testing of 2WS vehicles without and with power assistance as well as 4WS cars. The module of steering system mechanism model (p.4), as well as the modules of power assistance model and steer of rear wheels model are described with details in the monograph [14].

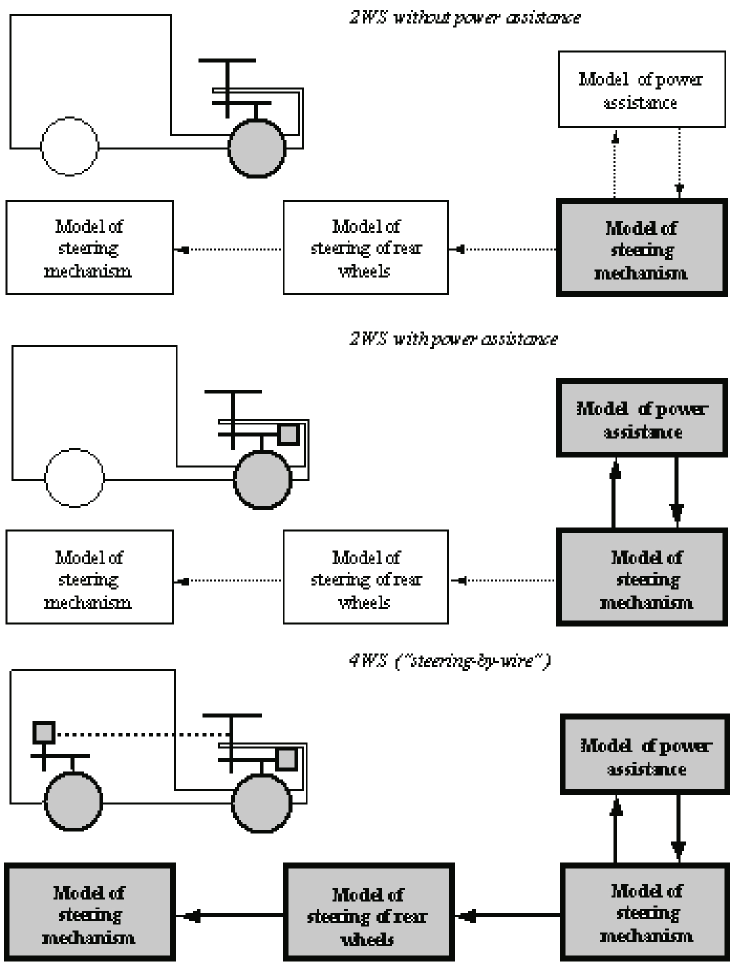

Fig. 7. Idea of modular form of steering system model

The FORS program enables:

- standard simulation (for one data set),
- sensitivity computation (comparative simulations with variation of parameter),

– bifurcation computation (comparative simulations for many values of parameter with registration output signals at selected moment of the time).

Example results of simulation computation for sensitivity analysis (Fig. 8) concern a combined ISO road test. The sequence of the test is: ISO7401 – ramp input on the steering wheel, then ISO4138 – circular steady state motion, an finally steering wheel released. The vehicle speed is 80 km/h.

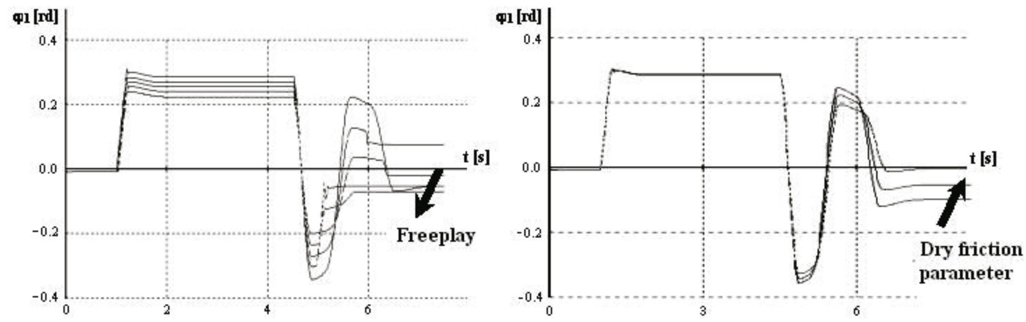

Fig. 8. Example simulation results (time history of left wheel angle) for combined ISO road test of 2WS car with steering system without power assistance. Dry friction parameter $M_{T\varphi 0} \in [1.35, 4.05]$Nm, freeplay parameter $(\delta - p\gamma)_0 \in [0, 0.27]$ rd

An extensive simulation data are easy to get in other author's papers (listed in [5] and [14]). On the base of those simulation investigations some conclusions on sensitivity of steering system model including friction/freeplay have been formulated.

The main conclusion is that the freeplay/friction parameters influence steering system dynamics and car steerability properties. This is evidently visible in simulation signals. But this influence is very complicated and depends of the car steering system structure (2WS without and with power assistance, 4WS). Introduction of the servomechanisms (2WS with power assistance) significantly extinguished temporary processes and a little diminished the sensitivity on the changes of the freeplay/friction parameters. Addition of the steering for the back wheels diminishes the sensitivity a little. Changing of the friction force characteristics (for example difference between maximum static friction force and kinetic dry friction force, or so called the Stribeck effect as a supplement to the Coulomb characteristics) does not influent considerable on angle motions of steered wheels.

6. Final remarks

Investigations concerning the friction or freeplay influence on the car steering system dynamics and the car steerability properties should be continued. This seems to be especially important for synthesis of so called robust control subsystems – components of mechatronic devices in the steering system and for synthesis of driver's assistance subsystems. Of course, extensive sensitivity studies of non-smooth steering system models can be very interesting for all researchers working on multi-body systems simulation software, for specialists of robotics and so on.

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